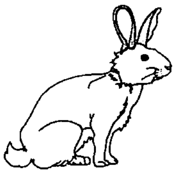
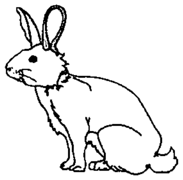
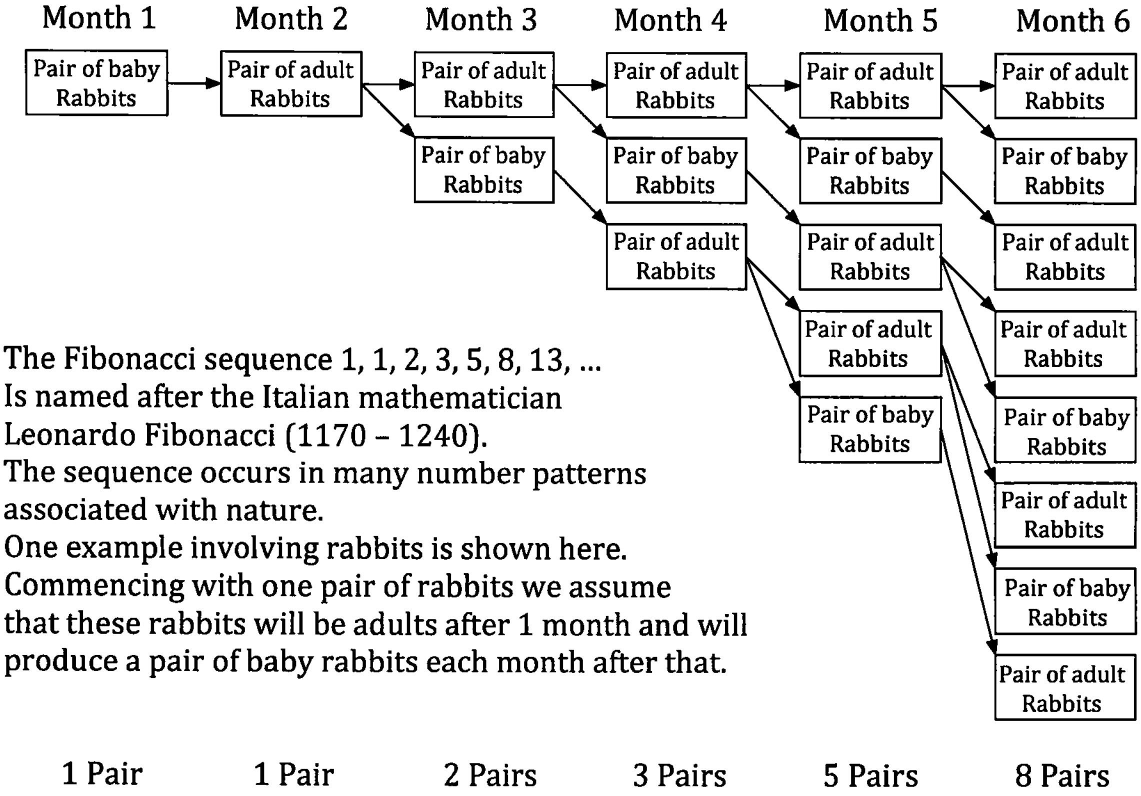
Chapter 3.

Sequences.



Situation One: Rabbits.





The Fibonacci sequence 1, 1, 2, 3, 5, 8, 13, ...
Is named after the Italian mathematician
Leonardo Fibonacci (1170 - 1240).
The sequence occurs in many number patterns
associated with nature.
One example involving rabbits is shown here.
Commencing with one pair of rabbits we assume
that these rabbits will be adults after 1 month and will
produce a pair of baby rabbits each month after that.

Notice that in the number sequence 1, 1, 2, 3, 5, 8, 13, ...
each number, after the first two ones, is the sum of the previous two numbers.

$$
\begin{aligned}
1 + 1 &= 2, \\
1 + 2 &= 3, \\
2 + 3 &= 5, \\
3 + 5 &= 8, \\
5 + 8 &= 13.
\end{aligned}
$$

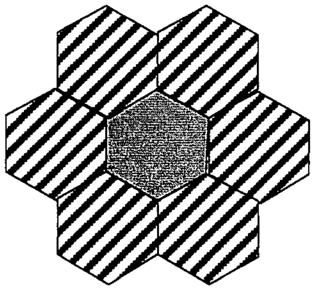
✍ Write the next five numbers in this sequence.

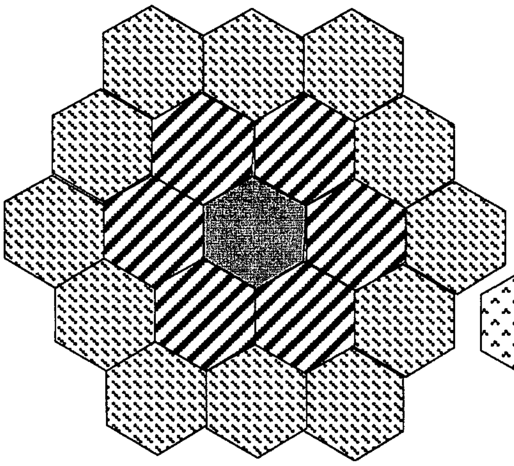
💻 Use the internet to investigate real world applications of the Fibonacci sequence of numbers.

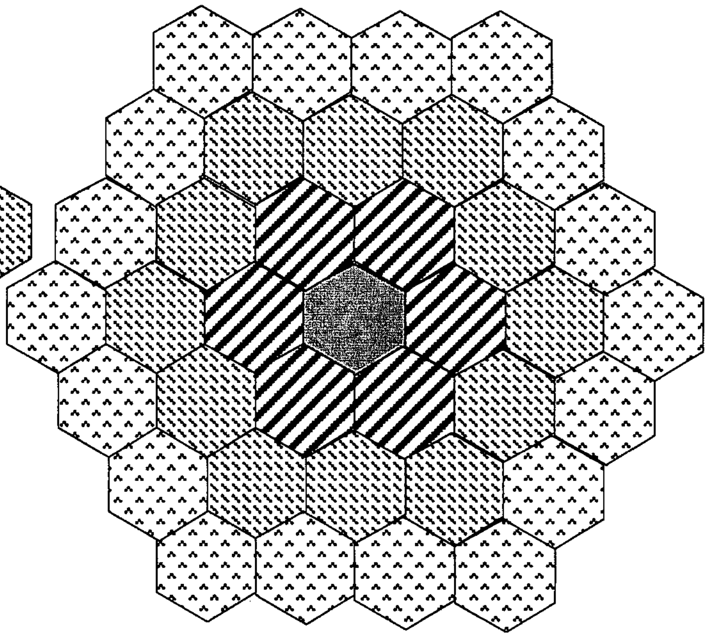
Situation Two:
Hexagonal patchworks.

Patchwork One shown on the right has a central hexagon and a ring of hexagons around it.

Placing a second ring of hexagons around Patchwork One gives *Patchwork Two* shown below left and a further ring of hexagons around Patchwork Two gives *Patchwork Three* shown below right.







For the three patchworks shown above, consider things like:

- the number of hexagons in the outer ring of each patchwork,
- the number of hexagonal pieces in each patchwork,
- the number of edges that form the border of each patchwork,
- other aspects you may think of,

and use your answers to try to predict what numbers the next few patchworks in the continuing pattern would have for these things.

Sequences.

Consider the following table of values for the linear function $y = 3x + 2$.

x	2	3	4	5	6	7	8
y	8	11	14	17	20	23	26

All pairs (x, y) in the table fit the rule $\quad y = 3x + 2$

For example, for (2, 8) $\quad 8 = 3(2) + 2$

for (3, 11) $\quad 11 = 3(3) + 2$

for (4, 14) $\quad 14 = 3(4) + 2$ etc.

However let us now just consider the **sequence** of y-values, i.e.

8 11 14 17 20 23 26

Note

- A **sequence** is a set of items belonging in a certain order, according to some rule. Knowing sufficient items, and the rule that is involved, the next item can be determined.
- In this course we will concentrate on <u>number</u> sequences.
- We refer to each number in the sequence as a **term** of the sequence. Thus in the above sequence the first term is 8, the second term is 11, the third term is 14, the fourth term is 17 and so on.
- Writing T_1 for the first term, T_2 for the second term and so on we write:

  $T_1 = 8 \quad T_2 = 11 \quad T_3 = 14 \quad T_4 = 17 \quad T_5 = 20 \quad T_6 = 23 \quad T_7 = 26$
- The situations at the beginning of this chapter involved sequences of numbers, for example the Fibonacci sequence:

  1 1 2 3 5 8 13 ...
- Whilst we usually use $T_1, T_2, T_3, T_4, T_5, \ldots$ (or perhaps $t_1, t_2, t_3, t_4, t_5, \ldots$) for the terms of a sequence other letters may be used at times. For example, for the terms of the Fibonacci sequence, we might use $F_1, F_2, F_3, F_4, F_5, \ldots$. Some calculators may use U or a, or some other letter.

Example 1

For the sequence 3, 10, 31, 94, 283, 850, 2 551, ... determine

(a) T_3 (b) T_5 (c) $T_3 + T_5$ (d) $3T_2$ (e) $2T_3$ (f) T_8

(a) T_3 is the 3rd term in the sequence. Thus $T_3 = 31$.

(b) T_5 is the 5th term in the sequence. Thus $T_5 = 283$.

(c) $T_3 + T_5 = 31 + 283$
$= 314$

(d) $3T_2 = 3(10)$
$= 30$

(e) $2T_3 = 2(31)$
$= 62$

(f) Noticing that each term is obtained by multiplying the previous term by 3 and then adding 1 it follows that

$$\begin{aligned} T_8 &= 3(T_7) + 1 \\ &= 3(2\,551) + 1 \\ &= 7\,654. \end{aligned}$$

Exercise 3A

For the sequence 10, 14, 18, 22, 26, 30, 34, 38, ... determine

1. T_3 2. T_5 3. $T_3 + T_5$ 4. T_8

5. $3T_2$ 6. $2T_3$ 7. $3(T_1 + T_2)$ 8. $3T_1 + T_2$

9. T_9 10. T_{10} 11. $T_3^{\ 2}$ 12. $T_2^{\ 3}$

For the sequence 5, 8, 11, 14, 17, 20, 23, ... determine

13. T_2 14. T_6 15. $T_2 + T_6$ 16. T_8

17. T_9 18. $T_3 + 2T_1$ 19. $T_1 + 2T_3$ 20. $(T_3 - T_2)^2$

For the sequence 2, 6, 18, 54, 162, 486, 1 458, ... determine

21. T_5 22. $3T_2$ 23. $T_1 + T_2 + T_3$ 24. T_8

Using C_n for the nth term of the cubic numbers 1, 8, 27, 64, 125, ... determine

25. C_3 26. C_6 27. C_7 28. $C_6 - C_5$

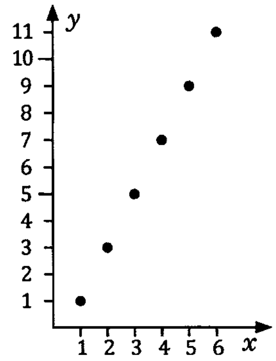
The Lucas sequence follows the same rule as the Fibonacci sequence, i.e. each term after the first two is the sum of the previous two terms. Using L_n for the nth term of the Lucas sequence and given that $L_1 = 1$ and $L_2 = 3$, determine

29. L_3 30. L_4 31. $L_4^{\ 2}$ 32. $2L_8$

Arithmetic sequences.

x	1	2	3	4	5	6
y	1	3	5	7	9	11

Notice that in the table of values on the right, as the x-values increase by 1 the y-values increase by 2. As we would expect from this constant first difference pattern of 2, graphing gives points that lie in a **straight line** and the gradient of the line is 2.



Sequences of numbers in which each term is obtained from the previous term by the addition of some constant number are said to be

Arithmetic Sequences,
Arithmetic Progressions
or simply *APs.*

For example 1, 2, 3, 4, 5, 6, ...
In this case we say that **the first term** is 1 and **the common difference** is 1.

Similarly for the AP: 1, 3, 5, 7, 9, 11, ...
we say that the first term is 1 and the common difference is 2.

For 7, 11, 15, 19, 23, 27, ...
the first term is 7 and the common difference is 4.

For 67, 62, 57, 52, 47, 42, ...
the first term is 67 and the common difference is -5.

Thus all arithmetic sequences are of the form:

$$a, \quad a + d, \quad a + 2d, \quad a + 3d, \quad a + 4d, \quad a + 5d, \quad a + 6d, \quad \ldots$$

In this general form we have a first term of "a" and common difference "d".

Consider again the arithmetic sequence:

$$1, \quad 3, \quad 5, \quad 7, \quad 9, \quad 11, \quad \ldots$$

Each term is obtained by adding 2 to the previous term. Thus if T_n is the n^{th} term it follows that the next term, T_{n+1}, will be $T_n + 2$.

Hence for this sequence $\quad T_{n+1} = T_n + 2, \quad$ (or $\quad t_{n+1} = t_n + 2$).

For the general arithmetic sequence at the top of the page

$$T_{n+1} = T_n + d, \qquad (\text{or} \quad t_{n+1} = t_n + d).$$

This rule tells us how the terms of the sequence *recur*. It is the **recursive rule** or **recursive equation** for the sequence. Given this rule and one term, usually the first, all other terms of the sequence can be determined.

Example 2

A sequence is such that $T_{n+1} = T_n + 5$ and the first term, T_1, is 7.

Find the first four terms of the sequence.

The recursive definition informs us that each term is the previous term add 5.

Hence if $T_1 = 7$ it follows that

$$\begin{aligned} T_2 &= 7 + 5 \\ &= 12 \\ T_3 &= 12 + 5 \\ &= 17 \\ T_4 &= 17 + 5 \\ &= 22 \end{aligned}$$

The first four terms of the sequence are 7, 12, 17, 22.

Example 3

For each of the following sequences state whether the sequence is an AP or not and, for those that are, state the first term, the common difference and a recursive formula.

(a) 13, 18, 23, 28, 33, 38, ...,

(b) 3, 6, 12, 24, 48, 96, ...,

(c) 90, 79, 68, 57, 46, 35, ...,

(a) Each term is 5 more than the previous term.

Thus the sequence <u>is</u> an Arithmetic Progression with first term = 13

the common difference = 5

and $T_{n+1} = T_n + 5$

(b) The terms do not have a common difference.

Thus the sequence is <u>not</u> an Arithmetic Progression.

(c) Each term is 11 less than the previous term.

Thus the sequence is an Arithmetic Progression with first term = 90

the common difference = −11

and $T_{n+1} = T_n - 11$

Note • The recursive rule from part (a) in the previous example, i.e.

$$T_{n+1} = T_n + 5$$

could equally well be written in the form

$$T_n = T_{n-1} + 5.$$

Both expressions tell us the same thing, i.e. that each term of the sequence is obtained by adding 5 to the previous term.

Example 4

A sequence is defined by $T_n = 3T_{n-1} - 2$ with $T_1 = 5$. Determine the first five terms of this sequence and hence determine whether the sequence is an arithmetic sequence or not.

The formula $T_n = 3T_{n-1} - 2$ tells us that each term is obtained by multiplying the previous term by 3 and then subtracting 2.

Thus if $T_1 = 5$ it follows that

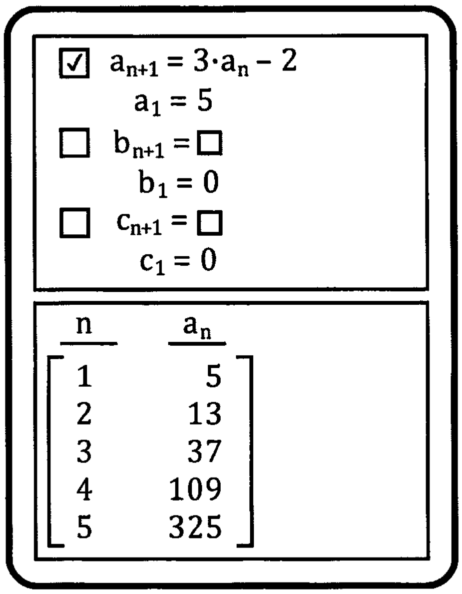
$$\begin{aligned} T_2 &= 3(5) - 2 = 13, \\ T_3 &= 3(13) - 2 = 37, \\ T_4 &= 3(37) - 2 = 109, \\ T_5 &= 3(109) - 2 = 325. \end{aligned}$$

The first five terms are 5, 13, 37, 109, 325.

These terms do <u>not</u> have a common difference. The sequence is <u>not</u> an arithmetic sequence.

$$\begin{array}{ccccccccc} 5 & & 13 & & 37 & & 109 & & 325 \\ & 8 & & 24 & & 72 & & 216 & \end{array}$$

Alternatively, for the previous example, a calculator could be used to display the terms of the sequence, once an appropriate recursive rule and first term have been entered.



☑ $a_{n+1} = 3\cdot a_n - 2$

$a_1 = 5$

☐ $b_{n+1} =$ ☐

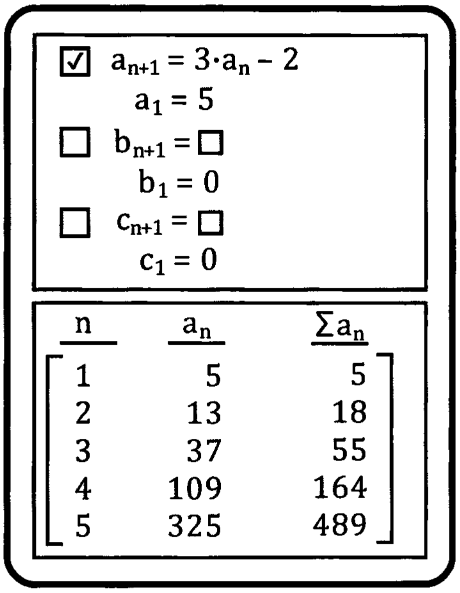
$b_1 = 0$

☐ $c_{n+1} =$ ☐

$c_1 = 0$

n	a_n
1	5
2	13
3	37
4	109
5	325

Notice that the display on the right is similar to that shown on the previous page but now the progressive sums (also called partial sums) of the terms are also displayed.



☑ $a_{n+1} = 3\cdot a_n - 2$
$a_1 = 5$
☐ $b_{n+1} =$ ☐
$b_1 = 0$
☐ $c_{n+1} =$ ☐
$c_1 = 0$

n	a_n	Σa_n
1	5	5
2	13	18
3	37	55
4	109	164
5	325	489

$T_1 = 5$

$T_1 + T_2 = 5 + 13 = 18$

$T_1 + T_2 + T_3 = 5 + 13 + 37 = 55$

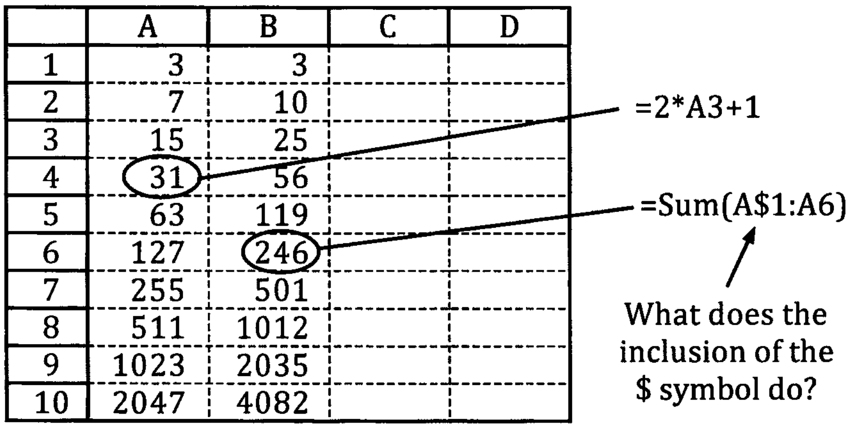
$T_1 + T_2 + T_3 + T_4 = 5 + 13 + 37 + 109 = 164$

$T_1 + T_2 + T_3 + T_4 + T_5 = 5 + 13 + 37 + 109 + 325 = 489$

Calculators differ in the way they accept and display such information. If you wish to use a calculator in this way make sure that **you** can use **your** calculator to input recursive formulae and to display the terms of a sequence.

Spreadsheets are another way of displaying the terms of a recursively defined sequence as shown below for the sequence with recursive definition

$$T_{n+1} = 2T_n + 1 \text{ with } T_1 = 3$$



	A	B	C	D
1	3	3		
2	7	10		
3	15	25		
4	31	56		
5	63	119		
6	127	246		
7	255	501		
8	511	1012		
9	1023	2035		
10	2047	4082		

=2*A3+1

=Sum(A$1:A6)

What does the inclusion of the $ symbol do?

Note that this sequence is **not** an arithmetic sequence – the entries in column A do not display a common difference pattern.

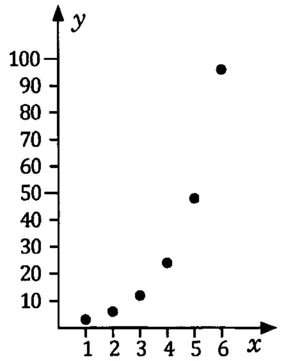
Once again the progressive sums (or partial sums) can easily be shown, as in column B.

Create the spreadsheet yourself and use the "fill down" ability when creating it.

Geometric sequences.

Notice that in the table of values on the right, as the x-values increase by 1 the y-values multiply by 2. As we would expect from this constant ratio of successive y-values, graphing gives the characteristic shape of an **exponential** function.

x	1	2	3	4	5	6
y	3	6	12	24	48	96



Sequences that progress by each term being obtained by multiplying the previous term by a constant number are said to be

Geometric Sequences,
Geometric Progressions
or simply GPs.

For example 5, 15, 45, 135, 405, 1215, ...
In this case we say that **the first term** is 5 and **the common ratio** is 3.

Similarly for the GP: 0·5, 1, 2, 4, 8, 16, ...
we say that the first term is 0·5 and the common ratio is 2.

For 1000, 100, 10, 1, 0·1, 0·01, ...
the first term is 1000 and the common ratio is 0·1.

For 64, 96, 144, 216, 324, 486, ...
the first term is 64 and the common ratio is 1·5.

Thus all GPs are of the form:

$$a, \quad ar, \quad ar^2, \quad ar^3, \quad ar^4, \quad ar^5, \quad ar^6, \quad \ldots$$

In this general form we have a first term of "a" and common ratio "r".

Using recursive notation we have $T_{n+1} = r \times T_n$ with $T_1 = a$, (or $t_{n+1} = r\,t_n$, $t_1 = a$).

Example 5

For each of the following sequences state whether the sequence is a geometric sequence or not and, for those that are, state the first term, the common ratio and a recursive formula.

(a) 3, 6, 12, 24, 48, 96, ...,
(b) 128, 96, 72, 54, 40·5, 30·375, ...,
(c) 4, 9, 14, 19, 24, 29, ...,

(a) $\frac{6}{3} = 2, \quad \frac{12}{6} = 2, \quad \frac{24}{12} = 2, \quad \frac{48}{24} = 2, \quad \frac{96}{48} = 2.$

Each term is the previous term multiplied by 2.

Thus the sequence <u>is</u> a geometric sequence with first term = 3
the common ratio = 2
and $T_{n+1} = 2T_n$
(or $T_n = 2T_{n-1}$)

(b) $\frac{96}{128} = 0{\cdot}75,\ \frac{72}{96} = 0{\cdot}75,\ \frac{54}{72} = 0{\cdot}75,\ \frac{40{\cdot}5}{54} = 0{\cdot}75,\ \frac{30{\cdot}375}{40{\cdot}5} = 0{\cdot}75.$

Each term is the previous term multiplied by 0·75.

Thus the sequence <u>is</u> a geometric sequence with first term = 128
the common ratio = 0·75
and $T_n = 0{\cdot}75T_{n-1}$

(c) $\frac{9}{4} = 2{\cdot}25,\ \frac{14}{9} = 1{\cdot}\overline{5}$.

The sequence does not have a common ratio.

Thus the sequence is <u>not</u> a geometric sequence.

Example 6

$400 is invested in an account and earns $20 interest each year.

(a) How much is the account worth after 1 year, 2 years, 3 years and 4 years?

(b) Do the amounts the account is worth at the end of each year form an arithmetic sequence, a geometric sequence or neither of these?

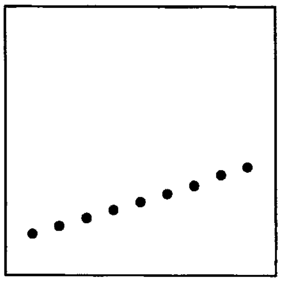
(a)

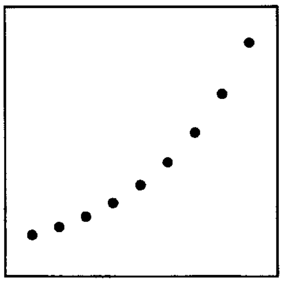
Initial value	value after 1 year	value after 2 years	value after 3 years	value after 4 years	...
$400	$400 + 1($20)	$400 + 2($20)	$400 + 3($20)	$400 + 4($20)	...
$400	$420	$440	$460	$480	...

After 1, 2, 3 and 4 years the account is worth $420, $440, $460 and $480 respectively.

(b) The situation gives rise to amounts with a common difference of $20.
The amounts the account is worth at the end of each year form an arithmetic sequence.

As you are probably aware, the situation described in the previous example is not the way that an investment usually earns interest. Once the $20 interest has been added at the end of year 1 the account has $420 in it and it is this $420 that attracts interest in year 2, not just the initial $400. In this way the interest earned in one year itself attracts interest in subsequent years, i.e. compound interest is involved, rather than the simple interest situation described in the previous example.





Example 7

\$2000 is invested and accrues interest at a rate of 10% per annum, compounded annually.

(a) If no further deposits are made how much will be in the account after 1 year, 2 years, 3 years and 4 years?

(b) Do the amounts the account is worth at the end of each year form an arithmetic sequence, a geometric sequence or neither of these?

(a)

Initial value	value after 1 year	value after 2 years	value after 3 years	value after 4 years	...
\$2000	$\$2000 \times 1{\cdot}1$	$\$2000 \times 1{\cdot}1^2$	$\$2000 \times 1{\cdot}1^3$	$\$2000 \times 1{\cdot}1^4$	...
\$2000	\$2200	\$2420	\$2662	\$2928·20	...

After 1, 2, 3 and 4 years the account is worth \$2200, \$2420, \$2662 and \$2928·20 respectively.

(b) The situation gives rise to amounts with a common ratio of 1·1.
The amounts the account is worth at the end of each year form a geometric sequence.

Exercise 3B

For each of the following arithmetic sequences state

- the first term, T_1,

and

- the $(n + 1)^{th}$ term, T_{n+1}, in terms of the n^{th} term, T_n.

1. 6, 10, 14, 18, 22, 26, ...
2. 28, 26, 24, 22, 20, 18, ...
3. 5, 15, 25, 35, 45, 55, ...
4. 7·5, 10, 12·5, 15, 17·5, 20, ...
5. 100, 89, 78, 67, 56, 45, ...

For each of the following geometric sequences state

- the first term, T_1,

and

- the n^{th} term, T_n, in terms of the $(n - 1)^{th}$ term, T_{n-1}.

6. 6, 12, 24, 48, 96, 192, ...
7. 0·375, 1·5, 6, 24, 96, 384, ...
8. 384, 96, 24, 6, 1·5, 0·375, ...
9. 50, 150, 450, 1350, 4050, 12150, ...
10. 1000, 1100, 1210, 1331, 1464·1, 1610·51, ...

For each of the following sequences state whether the sequence is arithmetic, geometric or neither of these two types.

11. 2, 5, 11, 23, 47, 95, ...
12. 1, 5, 25, 125, 625, 3125, ...
13. 13, 14·5, 16, 17·5, 19, 20·5, ...
14. 50, 39, 28, 17, 6, −5, ...
15. 1, 1, 2, 3, 5, 8, ...
16. 128, 160, 200, 250, 312·5, 390·625, ...
17. $T_{n+1} = 3T_n$, $T_1 = 3$.
18. $T_{n+1} = T_n + 6$, $T_1 = 2$.
19. $T_{n+1} = 3T_n + 5$, $T_1 = 1$.
20. $T_n = (T_{n-1})^2$, $T_1 = 7$.
21. $T_n = T_{n-1} - 8$, $T_1 = 2\,000$.
22. $T_n = (0{\cdot}5)T_{n-1}$, $T_1 = 8$.

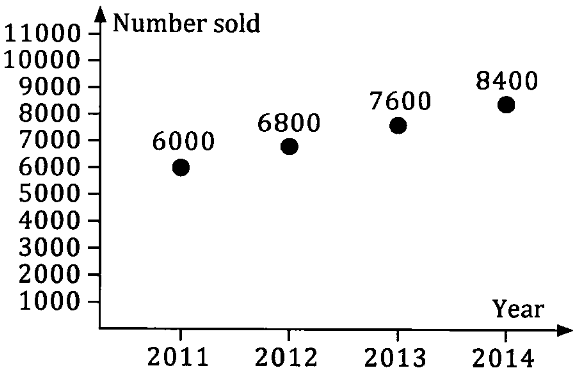
23. An AP has a first term of 8 and a common difference of 3. Determine the first four terms of the sequence and the recursive rule for T_{n+1} in terms of T_n.

24. An AP has a first term of 100 and a common difference of −3. Determine the first four terms of the sequence and the recursive rule for T_{n+1} in terms of T_n.

25. A GP has a first term of 11 and a common ratio of 2. Determine the first four terms of the sequence and the recursive rule for T_{n+1} in terms of T_n.

26. A GP has a first term of 2048 and a common ratio of 0·5. Determine the first four terms of the sequence and the recursive rule for T_{n+1} in terms of T_n.

27. The graph on the right shows the number of vehicles a company sold in a particular country each year from 2011 to 2014.
    (a) Verify that the figures for these years are in arithmetic progression.
    (b) With $N_{2011} = 6000$ write a recursive rule for N_{n+1} in terms of N_n.



28. Each term of a sequence is obtained using the recursive rule

$$T_n = T_{n-1} + 10\% \text{ of } T_{n-1}$$

(a) Is the sequence an arithmetic progression, a geometric progression or neither of these?
(b) If the first term of the sequence is 500 find the next three terms.

29. Each term of a sequence is obtained using the recursive rule

$$T_n = T_{n-1} + 25\% \text{ of } T_{n-1}$$

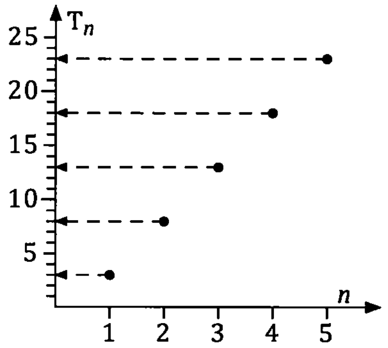
(a) Is the sequence an arithmetic progression, a geometric progression or neither of these?
(b) If the first term of the sequence is 1000 find the next three terms.

30. Each term of a sequence is obtained using the recursive rule

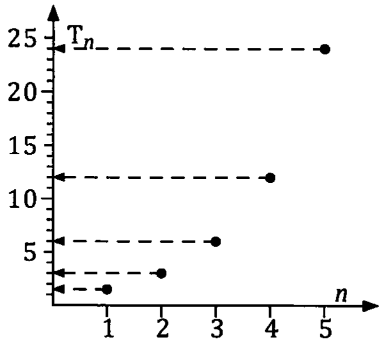
$$T_n = T_{n-1} - 10\% \text{ of } T_{n-1}$$

(a) Is the sequence an arithmetic progression, a geometric progression or neither of these?
(b) If the first term of the sequence is 24000 find the next three terms.

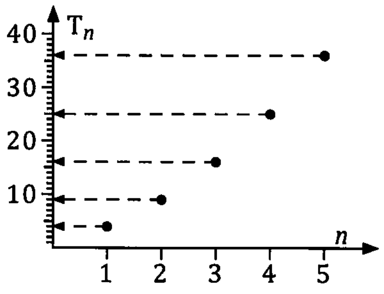
31. The graph on the right indicates the first five terms, T_1 to T_5, of a sequence, all of which are whole numbers.
State (a) the first term and a recursive rule for the sequence,
(b) whether the sequence is arithmetic, geometric or neither of these types.



32. The graph on the right indicates the first five terms, T_1 to T_5, of a sequence. The last four of these terms are whole numbers.
State (a) the first term and a recursive rule for the sequence,
(b) whether the sequence is arithmetic, geometric or neither of these types.



33. The graph on the right indicates the first five terms, T_1 to T_5, of a sequence. All of these terms are whole numbers.
Determine (a) the first five terms,
and (b) whether the sequence is arithmetic, geometric or neither of these types.



34. \$1200 is invested in an account and earns \$96 interest each year.
    (a) How much is the account worth after 1 year, 2 years, 3 years and 4 years?
    (b) Are these amounts in arithmetic progression, geometric progression or neither of these?
    (c) Express the sequence of values:
    Initial value, value after 1 year, value after 2 years, ...
    using recursive notation.

35. The number of stitches in each row of a particular crochet pattern are as shown on the right.
    With T_1 the number of stitches in the first row, T_2 the number in the second row etc, express the sequence

    $$T_1, T_2, T_3, T_4, T_5, \ldots$$

    using recursive notation (i.e. state T_1 and the recursive rule).

1st row ⊕ ⊕ ⊕ ⊕
2nd row ⊕ ⊕ ⊕ ⊕ ⊕
3rd row ⊕ ⊕ ⊕ ⊕ ⊕ ⊕
4th row ⊕ ⊕ ⊕ ⊕ ⊕ ⊕ ⊕

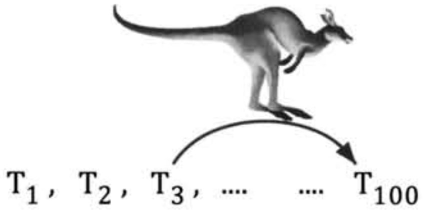
36. Won Yim starts working for a particular company on the 1st January one year and is paid an initial annual salary of \$45000 with a guaranteed \$1500 rise each year for the next 7 years.
    Express the sequence of annual salaries over this time as a sequence using recursive notation and state whether the terms of the sequence progress arithmetically, geometrically or neither of these.

37. Joe started a new job on 1st January 2014 and, during 2014, he received a salary of \$68 000. His contract guarantees a salary increase of 5% of the salary of the previous year on each subsequent 1st January, until and including 1st January 2017. Calculate Joe's salary for each year from 2014 to 2017.
    Express the sequence of salaries from 2014 (term one) to 2017 (term four) using recursive notation.

38. \$1500 is invested and accrues interest at a rate of 8% per annum, compounded annually. With this \$1 500 as the first term in the sequence express the value of the account on this and each subsequent year as a sequence defined recursively.

39. Each year the value of a car depreciates by 15% of its value at the beginning of that year. The car is initially worth \$36 000. With this \$36000 as the first term express the value of the car on this and each subsequent year as a sequence defined recursively.

Jumping to later terms of arithmetic and geometric sequences.



Consider the arithmetic sequence defined by

$$T_{n+1} = T_n + 2 \text{ and } T_1 = 3$$

The rule allows us to obtain the terms of the sequence:

$$\begin{aligned} T_1 &= 3 \\ T_2 &= T_1 + 2 = 3 + 2 = 5 \\ T_3 &= T_2 + 2 = 5 + 2 = 7 \\ T_4 &= T_3 + 2 = 7 + 2 = 9 \qquad \text{etc.} \end{aligned}$$

However, if we wanted to know the value of a term much later in the sequence, say T_{100}, it would be a tedious process to have to calculate all of the terms up to T_{100}. It would be more useful if we could jump to the desired term without having to determine all of the preceding ones.

Example 8

For the sequence defined recursively as $T_{n+1} = T_n + 7$ with $T_1 = 25$, determine the first four terms and the one hundredth term.

With $T_{n+1} = T_n + 7$ it follows that

$$\begin{aligned} T_2 &= T_1 + 7 \\ &= 32. \\ T_3 &= T_2 + 7 \\ &= 39. \\ T_4 &= T_3 + 7 \\ &= 46. \end{aligned}$$

Noticing that by the second term we have added 7 *once*, by the third time we have added 7 *twice*, by the fourth term we have added 7 *three* times. It follows that for the one hundredth term we need to have added 7 ninety nine times.

Hence

$$\begin{aligned} T_{100} &= T_1 + 99(7) \\ &= 25 + 693 \\ &= 718. \end{aligned}$$

The first four terms are 25, 32, 39 and 46 and the one hundredth term is 718.

If we apply the thinking used in the previous example to the general arithmetic sequence

$$a, \quad a + d, \quad a + 2d, \quad a + 3d, \quad a + 4d, \quad a + 5d, \quad \ldots$$

we note that for T_2 the common difference, d, has been added once, for T_3 it has been added twice, for T_4 it has been added three times, and so on. Thus for T_n we need to add the common difference (n – 1) times.

Thus the Arithmetic Progression

$$a, \quad a + d, \quad a + 2d, \quad a + 3d, \quad a + 4d, \quad a + 5d, \quad \ldots$$

has an nth term given by:

$$\boxed{\mathbf{T_n = a + (n - 1)\,d}} \qquad \text{Or} \quad t_n = t_1 + (n - 1)\,d$$

Just pause for a moment and check that you understand the difference between a recursive formula, which tells you how each term is obtained from the previous term, and the formula for the nth term, which allows you to determine any term.

Note • Thinking of T_n as y, and n as x, the reader should see similarities between

$$T_n = a + (n - 1)\,d \qquad \text{and} \qquad y = mx + c.$$

This is no surprise given the linear nature of arithmetic sequences.

Example 9

For the AP: 11, 14, 17, 20, ...

Determine (a) T_{123},

(b) T_{500},

(c) which term of the sequence is the first to exceed 1 000 000.

(a) $T_{123} = 11 + 122\,(3)$

$= 377$

(b) $T_{500} = 11 + 499\,(3)$

$= 1508$

(c) Suppose that T_n is the first term to exceed 1 000 000.

Now $T_n = 11 + (n - 1)\,3$

$\therefore \quad 11 + (n - 1)\,3 > 1000000$

i.e. $n > 333330{\cdot}\overline{6}$

Thus the first term to exceed 1 000 000 is $T_{333\,331}$.

Example 10

An AP has a 50th term of 209 and a 61st term of 253. Find (a) the 62nd term,
(b) the 1st term.

(a) To go from T_{50} to T_{61} we must add the common difference 11 times.

Thus if d is the common difference then

$$\begin{aligned} 253 - 209 &= 11d \\ \therefore \quad 44 &= 11d \\ \text{giving} \quad d &= 4 \\ \text{Hence} \quad T_{62} &= T_{61} + 4 \\ &= 257 \end{aligned}$$

The 62nd term is 257.

(b) From our understanding of APs it follows that

$$\begin{aligned} T_{50} &= T_1 + 49d \\ \therefore \quad 209 &= T_1 + 49(4) \\ 209 - 196 &= T_1 \\ T_1 &= 13 \end{aligned}$$

The 1st term is 13.

Alternatively we could use the given information to write

$$\begin{aligned} a + 49d &= 209 \\ \text{and} \quad a + 60d &= 253 \end{aligned}$$

and then solve these equations simultaneously.

Example 11

For the sequence defined recursively as $T_{n+1} = 1{\cdot}5T_n$ with $T_1 = 8\,192$, determine the first four terms and the fifteenth term.

With $T_{n+1} = 1{\cdot}5T_n$ it follows that

$$\begin{aligned} T_2 &= 1{\cdot}5T_1 \\ &= 8\,192 \times 1{\cdot}5 \\ &= 12\,288 \\ T_3 &= 1{\cdot}5T_2 \\ &= 12\,288 \times 1{\cdot}5 \\ &= 18\,432 \\ T_4 &= 1{\cdot}5T_3 \\ &= 18\,432 \times 1{\cdot}5 \\ &= 27\,648 \end{aligned}$$

Note that for the second term we multiply by 1·5 *once*, for the third term we multiply by 1·5 *twice*, for the fourth term we multiply by 1·5 *three* times. It follows that for the fifteenth term we need to multiply by 1·5 fourteen times.

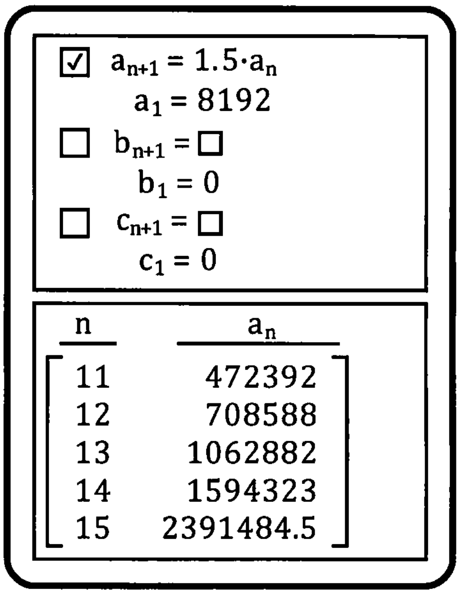
$$\begin{aligned} \text{Hence} \quad T_{15} &= T_1 \times 1{\cdot}5^{14} \\ &= 8\,192 \times 1{\cdot}5^{14} \\ &= 2\,391\,484{\cdot}5 \end{aligned}$$

The first four terms are 8 192, 12 288, 18 432 and 27 648.

The fifteenth term is 2 391 484·5.

Make sure you can obtain this same value for the fifteenth term of the sequence of the previous example by using a calculator to display terms of the sequence.

Are the larger numbers displayed as shown on the right or does your calculator use scientific notation to display these numbers?



☑ $a_{n+1} = 1.5 \cdot a_n$

$a_1 = 8192$

☐ $b_{n+1} =$ ☐

$b_1 = 0$

☐ $c_{n+1} =$ ☐

$c_1 = 0$

n	a_n
11	472392
12	708588
13	1062882
14	1594323
15	2391484.5

If we apply the thinking of the previous example to the general geometric sequence:

$$a, \quad ar, \quad ar^2, \quad ar^3, \quad ar^4, \quad ar^5, \quad ar^6, \quad \ldots$$

we note that T_2 is ar^1, T_3 is ar^2, T_4 is ar^3 etc. Thus $T_n = ar^{n-1}$.

Thus the Geometric Progression

$$a, \quad ar, \quad ar^2, \quad ar^3, \quad ar^4, \quad ar^5, \quad ar^6, \quad \ldots$$

has an nth term given by:

$$\boxed{\mathbf{T_n = a \times r^{n-1}}} \qquad \text{Or} \quad t_n = t_1 r^{n-1}$$

Note
- Thinking of T_n as y, and n as x, the reader should see similarities between

$$T_n = a \times r^{n-1} \qquad \text{and} \qquad y = k \times b^x.$$

Again no surprise given the exponential nature of geometric sequences.

Example 12

Determine the 12th term and the 15th term of the geometric sequence:

0·0025, 0·01, 0·04, 0·16, ...

By inspection the common ratio is 4.

Hence the 12th term will be $0{\cdot}0025 \times 4^{11}$

$= 10\,485{\cdot}76$

and the 15th term will be $0{\cdot}0025 \times 4^{14}$

$= 671\,088{\cdot}64$

Again make sure that you can obtain these same answers using the ability of some calculators to display the terms of a sequence.

Example 13

The 13th term of a GP is 12 288 and the 16th term is 98 304. Find (a) the 17th term,
(b) the 1st term.

(a) To go from the 13th term to the 16th term we must multiply by the common ratio 3 times.

If r is the common ration then $T_{16} = T_{13} \times r^3$

$\therefore$ $98304 = 12288 \times r^3$

Giving $r = 2$

Hence $T_{17} = T_{16} \times 2$

$= 196608$

The 17th term is 196 608.

(b) From our understanding of GPs it follows that $T_{13} = T_1 \times r^{12}$

i.e. $12288 = T_1 \times 2^{12}$

Giving $T_1 = 3$

The 1st term is 3.

(Alternatively we could write $ar^{12} = 12288$ and $ar^{15} = 98304$ and solve simultaneously.)

Growth and decay – again!

Consider the growth in the value of a house that is initially valued at $500 000 and is subject to an annual increase in value of 6·4%.

Initial value = $500 000 ← T_1
Value after 1 year = $500 000 ×1·064 ← T_2
Value after 2 years = $500 000 ×$1{\cdot}064^2$ ← T_3
Value after 3 years = $500 000 ×$1{\cdot}064^3$ ← T_4 etc.

These values form a geometric sequence with $T_{n+1} = T_n \times 1{\cdot}064$
and $T_1 = 500000$

Asked a question like: *At this rate how many years will it take for the value of this house to reach a value of $1 000 000?*

We could use our ability to solve exponential equations, as covered in earlier chapters, and recognizing that after x years the value will be $500 000 × $1{\cdot}064^x$ simply ask a calculator to solve the equation

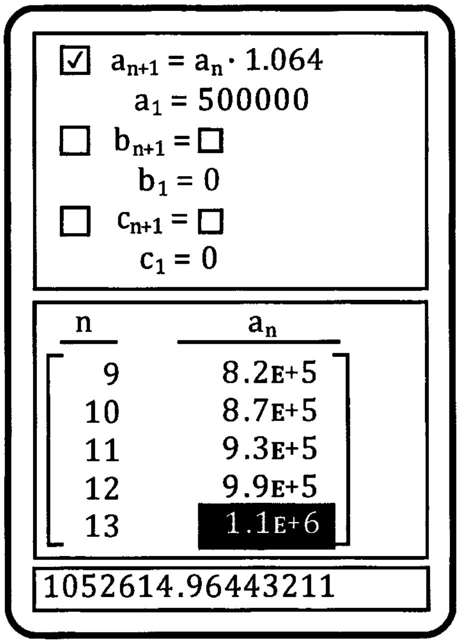
$$\$500000 \times 1{\cdot}064^x = \$1000000$$

To obtain the value $x = 11{\cdot}17$ (correct to 2 d.p.)

Hence the value of the house will be $1 000 000 shortly after the end of the 11th year, i.e. early in the 12th year.

Alternatively, if we wanted to see the progressive year by year values, we could display the terms of our sequence on a calculator or spreadsheet, as shown on the next page.

◆	A	B	C	D
1	Initial value			$500,000.00
2	Percentage increase			6.40
3	Value at end of year		1	$532,000.00
4			2	$566,048.00
5			3	$602,275.07
6			4	$640,820.68
7			5	$681,833.20
8			6	$725,470.52
9			7	$771,900.64
10			8	$821,302.28
11			9	$873,865.63
12			10	$929,793.03
13			11	$989,299.78
14			12	$1,052,614.96



☑ $a_{n+1} = a_n \cdot 1.064$

$a_1 = 500000$

☐ $b_{n+1} = \square$

$b_1 = 0$

☐ $c_{n+1} = \square$

$c_1 = 0$

n	a_n
9	8.2E+5
10	8.7E+5
11	9.3E+5
12	9.9E+5
13	1.1E+6

1052614.96443211

Create a spreadsheet like that shown above yourself.

As before, the value of the house will be $1 000 000 shortly after the end of the eleventh year, i.e. early in the 12th year.

However note carefully that in this situation, with the recursive definition

$$T_{n+1} = T_n \times 1{\cdot}064 \quad \text{and} \quad T_1 = 500000$$

T_1 is the value after **zero** years. Hence we must remember that if we use the ability of a calculator to generate the terms of the sequence, according to the recursive rule given above, then the balance after n years will be given by T_{n+1}. I.e. in the calculator display above, n = 13 gives the value at the end of 12 years.

One way to avoid this possible source of confusion would be to use the ability of some calculators to accept a sequence defined using T_0 as the 1st term.

I.e. define the sequence as:

$T_{n+1} = T_n \times 1{\cdot}064$ and $T_0 = 500000$,

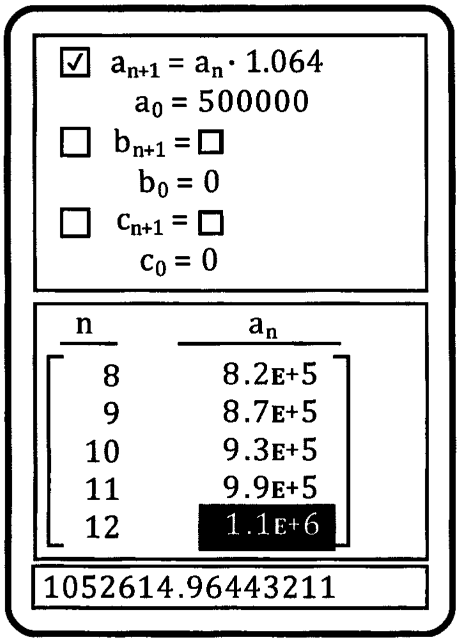
as shown on the right.

Under such a definition T_n would indeed be the value after n years.

Alternatively we could use

$T_{n+1} = T_n \times 1{\cdot}064$ and $T_1 = 500000 \times 1{\cdot}064$

and again T_n would be the value after n years.



☑ $a_{n+1} = a_n \cdot 1.064$

$a_0 = 500000$

☐ $b_{n+1} = \square$

$b_0 = 0$

☐ $c_{n+1} = \square$

$c_0 = 0$

n	a_n
8	8.2E+5
9	8.7E+5
10	9.3E+5
11	9.9E+5
12	1.1E+6

1052614.96443211

Exercise 3C

Without using the sequence display routine available on some calculators determine the one hundredth term in each of the following arithmetic sequences.

1. 11, 16, 21, 26, 31, 36, ...
2. −8, −5, −2, 1, 4, 7, ...
3. $T_{n+1} = T_n + 8$ with $T_1 = 23$.
4. $T_{n+1} = T_n - 2$ with $T_1 = 78$.

Without using the sequence display routine available on some calculators determine the twenty fifth term in each of the following geometric sequences, leaving your answers in the form $a \times b^n$.

5. 5, 10, 20, 40, 80, 160, ...
6. 1·5, 6, 24, 96, 384, 1536, ...
7. $T_{n+1} = 3T_n$ with $T_1 = 8$.
8. $T_{n+1} = 2T_n$ with $T_1 = 11$.

Use the ability of some calculators to display the terms of a sequence to determine the requested term in each of the following sequences.

9. $T_{n+1} = T_n + 8$ with $T_1 = 7$. Determine T_{28}.
10. $T_{n+1} = 35 - 2T_n$ with $T_1 = 5$. Determine T_{20}.
11. $T_{n+1} = 3T_n + 2$ with $T_1 = 1$. Determine T_{19}.
12. $T_{n+1} = (-1)^n T_n + 3$ with $T_1 = 6$. Determine T_{45}.

13. Julie starts a new job at a factory manufacturing automobile components. The machine she operates requires several weeks before the operator is fully accustomed to it and so her output increases each day for the first 3 weeks (15 days). On the first day she successfully completes 48 items on the machine and increases this by 3 each day after that up to and including her 15th day on the machine.
    Express the number of items completed on each of the first 15 days as a sequence using recursive notation.
    How many items does she successfully complete on this 15th day on the machine?

14. Use the formula for the n^{th} term of an AP with common difference d and $T_1 = a$, i.e. the formula $T_n = a + (n - 1)d$, to explain why that for this AP, when we plot T_n on the y-axis and n on the x-axis the points obtained lie on a straight line of gradient d, and find the coordinates of the point where this straight line cuts the y-axis.

15. Use the formula for the n^{th} term of an GP with common ratio r and $T_1 = a$ to explain why that for this GP, when we plot T_n on the y-axis and n on the x-axis the points obtained fit an exponential curve and find the equation of this curve and the coordinates of the point where it cuts the y-axis.

16. Write a few sentences explaining what happens to the terms of the following arithmetic progression as $n \rightarrow \infty$.
    $T_1 = a$, $T_2 = a + d$, $T_3 = a + 2d$, $T_4 = a + 3d$, ... $T_n = a + (n - 1)d$, ...

17. Write a few sentences explaining what happens to the terms of the following geometric progression as $n \rightarrow \infty$.
    $T_1 = a$, $T_2 = ar$, $T_3 = ar^2$, $T_4 = ar^3$, ... $T_n = ar^{n-1}$, ...

18. An arithmetic sequence has a first term of 8 and a common difference of 3. Determine the first four terms, the 50th term and the 100th term of the sequence.

19. An arithmetic sequence has a first term of 100 and a common difference of −3. Determine the first four terms, the 50th term and the 100th term of this sequence.

20. A geometric sequence has a first term of 11 and a common ratio of 2. Determine the first four terms, the 15th term and the 25th term of this sequence.

21. A geometric sequence has a first term of 2048 and a common ratio of 0·5. Determine the first four terms and the sixteenth term of this sequence.

22. Find an expression for T_n in terms of n for each of the following APs.
    (a) 9, 15, 21, 27, 33, ... (b) 7, 8·5, 10, 11·5, 13, ...

23. Find an expression for T_n in terms of n for each of the following GPs.
    (a) 3, 6, 12, 24, 48, ... (b) 100, 110, 121, 133·1, 146·41, ...

24. For the arithmetic sequence: 2, 9, 16, 23, ...
    Determine (a) T_{123}, (b) T_{500},
    (c) which term of the sequence is the first to exceed 1 000 000.

25. For the geometric sequence: 0·0026, 0·013, 0·065, 0·325, ...
    Determine (a) T_{12},
    (b) which term of the sequence is the first to exceed 1 000 000.

26. For the GP: 20 000 000, 15 000 000, 11 250 000, 8 437 500, ...
    Determine (a) T_{12}, giving your answer to the nearest hundred,
    (b) which term of the sequence is the first less than 1.

27. The nth term of a sequence is given by $T_n = n^3$.

    Obtain the first four terms of this sequence and state whether the sequence is an arithmetic, geometric or neither of these.

28. An arithmetic sequence has a 19th term of 61 and a 41st term of 127.
    Find (a) the 20th term, (b) the 1st term.

29. An arithmetic sequence has a 50th term of 1853 and a 70th term of 1793.
Find (a) the 51st term, (b) the 1st term.

30. A geometric sequence has a 10th term of 98 415 and a 13th term of 2 657 205.
Find (a) the 14th term, (b) the 1st term.

31. A geometric sequence has a 7th term of 28 672, a 9th term of 458 752 and a negative common ratio.
Find (a) the 10th term, (b) the 1st term.

Note for questions 32 and 33.

Do each of the following two questions twice.

First using an exponential function approach, as in previous chapters,

and then by formulating a recursive formula for a sequence and then viewing the terms of the sequence on a calculator or spreadsheet.

32. $4000 is invested into an account paying interest at 8%, compounded annually. Determine (to the nearest cent) the amount in the account at the end of ten years.

33. If a house currently valued at $600 000 were to gain in value at 5·6% per annum, compounded annually, when would its value first exceed two million dollars?

Note for questions 34 and 35.

The next two questions involve an initial amount being invested into an account paying interest and each year a further amount being added (question 34) or subtracted (question 35). The amounts in the account each year no longer progress geometrically but the questions can be solved using the ability of some calculators to display the terms of a sequence defined recursively.

34. $4000 is invested into an account paying interest at 8%, compounded annually and an extra $200 is invested after each 12 months. Thus:
Amount in account at end of 1 yr = \$4000 × 1·08 + \$200 ← T_1
Amount in account at end of 2 yrs = (\$4000 × 1·08 + \$200) × 1·08 + \$200 ← T_2
Express T_{n+1} in terms of T_n and determine (nearest cent) the amount in the account at the end of ten years, after the $200 for that year has been added.

35. $4000 is invested into an account paying interest at 8%, compounded annually and $200 is withdrawn from the account after each 12 months. Thus:
Amount in account at end of 1 yr = \$4000 × 1·08 – \$200 ← T_1
Amount in account at end of 2 yrs = (\$4000 × 1·08 – \$200) × 1·08 – \$200 ← T_2
Express T_{n+1} in terms of T_n and determine (nearest cent) the amount in the account at the end of ten years after the $200 for the year has been withdrawn.

Miscellaneous Exercise Three.

This miscellaneous exercise may include questions involving the work of this chapter, the work of any previous chapters, and the ideas mentioned in the preliminary work section at the beginning of the book.

1. For each of the following state whether the relationship between the variables x and y is linear, quadratic, exponential or reciprocal.

   (a) $y = x^2$
   (b) $y = 2^x$
   (c) $y = 5x - 7$
   (d) $y = x^2 - 3x + 4$
   (e) $y = \frac{4}{x}$
   (f) $y = \frac{x}{4}$
   (g) $y = 0{\cdot}5x - 12$
   (h) $y - 6x = x^2 + 7$
   (i) $y = (x - 5)(x + 4)$
   (j) $xy = 7$
   (k) $y + 8 = 2x$
   (l) $y = 6 \times 3^x$

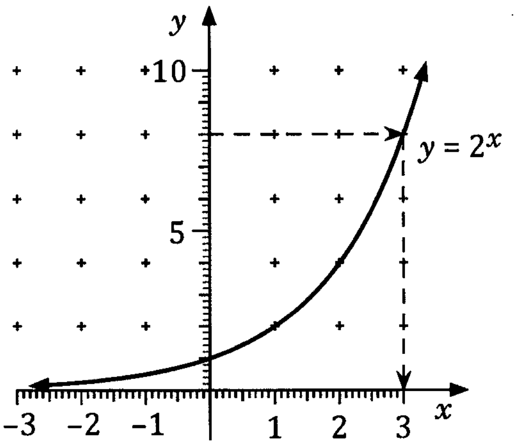
2. Our knowledge of the powers of 2 allows us to solve the equation $2^x = 8$ easily:

   $$x = 3.$$

   This answer is also evident from the graph of $y = 2^x$ shown on the right, if we find the x value for which y, and hence 2^x, equals 8.

   Use the graph to estimate solutions to the following equations:

   (a) $2^x = 4{\cdot}8$
   (b) $2^x = 6{\cdot}2$
   (c) $2^x = 2{\cdot}6$



3. If the following are all written in the form 2^n determine the value of n for each case.

   (a) 8
   (b) $\frac{1}{8}$
   (c) $\frac{1}{2}$
   (d) $\sqrt{2}$
   (e) 1
   (f) $\sqrt{8}$
   (g) $\frac{1}{64}$
   (h) $2\sqrt{2}$

4. Determine a formula for T_n, the nth term of a geometric sequence, for which

   $$T_2 = 6 \text{ and } T_5 = 20{\cdot}25,$$

   giving your answer (a) in the form $T_n = k \times r^{n-1}$,
   and also (b) in the form $T_n = k \times r^n$.

5. Find the 11th term of the geometric sequence that commences $1, \sqrt{3}, 3, \ldots$

6. Determine the value of x in each of the following:

(a) $4^x = 64$ (b) $4^x = \frac{1}{64}$ (c) $4^x = 0{\cdot}25$

(d) $64^{0{\cdot}5} = x$ (e) $x^2 = 64$ (f) $4^8 = 4^x \times 4^{-3}$

7. Evaluate each of the following without using a calculator.

(a) $16^{0{\cdot}5}$ (b) $16^{\frac{3}{2}}$ (c) $27^{\frac{2}{3}}$ (d) $25^{-0{\cdot}5}$ (e) $\left(\frac{1}{4}\right)^{-0{\cdot}5}$

8. A sequence has the recursive formula $T_{n+1} = (-1)^n\ T_n$, with $T_1 = 4$.
   - (a) Substitute n = 1 into the formula to determine T_2.
   - (b) Determine T_3 to T_5.
   - (c) Is the sequence arithmetic, geometric or neither of these?

9. A sequence has the recursive formula $T_{n+1} = (-1)^n\ 2T_n$, with $T_1 = 1$.
   - (a) Substitute n = 1 into the formula to determine T_2.
   - (b) Determine T_3 to T_5.
   - (c) Is the sequence arithmetic, geometric or neither of these?

10. An arithmetic sequence has a first term of 5k + 3, and a common difference of 5 – k.
    - (a) Find in terms of k an expression for the tenth term of the sequence, simplifying your answer where possible.
    - (b) If the 20th term of this sequence is 91 find the value of the 21st term.

For numbers 11 to 19 simplify each expression without the assistance of a calculator, expressing your answers in terms of positive indices.

11. $a^4 \times a^3$
12. $4x^2y \times 3xy^3$
13. $(15a^3b) \div (10ab^3)$
14. $(-3a)^2 \times (2a^2b)^3$
15. $\frac{(-3a)^2}{(2a^2b)^3}$
16. $\frac{6a^{-1}}{(8b)^{-1}}$
17. $\frac{6a^2b^{-4}}{3a^{-3}b}$
18. $\frac{k^7 + k^3}{k^3}$
19. $\frac{p^5 - p^8}{p^2}$

Without the assistance of a calculator, evaluate each of the following.

20. $\frac{5^{k+2}}{5^{k-1}}$
21. $\frac{5^{n+2} - 50}{5^n - 2}$
22. $\frac{2^{h+3} + 8}{3 \times 2^h + 3}$